Emmanuel College

MA 242 – Multivariable Calculus

Course Material

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The following is a summary of course material. Blue links lead to more detail; red links lead back.

topics

further content

vectors

"vector" (versus "scalar")

"component form" of a vector (in 2 or 3 dimensions)

vector operations and their properties

the standard bases The standard basis for $\mathbb{R}^2$ is $\mathbf{i} = (1,0)$, $\mathbf{j} = (0,1)$. The standard basis for $\mathbb{R}^3$ is $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$.

examples

point vs. vector

"length" The length of a vector $\langle x_1, x_2 \rangle \in \mathbb{R}^2$ is $\sqrt{x_1^2 + x_2^2}$. The length of a vector $\langle x_1, x_2, x_3 \rangle \in \mathbb{R}^3$ is $\sqrt{x_1^2 + x_2^2 + x_3^2}$.

fact

"unit vector" A vector $\mathbf{u}$ is called a unit vector if $|\mathbf{u}| = 1$. If $\mathbf{v} \neq \mathbf{0}$ is a vector $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the same direction as $\mathbf{v}$. The equation

$$\mathbf{v} = |\mathbf{v}| \frac{\mathbf{v}}{|\mathbf{v}|}$$

expresses $\mathbf{v}$ in terms of its length and direction.

"dot product" The dot product of $\mathbf{v} = (v_1, v_2), \mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ is

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2.$$

The dot product of $\mathbf{v} = (v_1, v_2, v_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$ is

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

theorem If $\theta \in [0, \pi)$ is the angle between two vectors $\mathbf{v}$ and $\mathbf{w}$, then

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos \theta.$$

proof

properties of the dot product

"orthogonal" Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be orthogonal if $\mathbf{v}\cdot\mathbf{w} = 0$. If $\mathbf{v}, \mathbf{w} \neq 0$, then this means that the angle between them is π. (That is, they are perpendicular.)

"parallel" Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be parallel if $\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|$. If $\mathbf{v}, \mathbf{w} \neq 0$, then this means that the angle between them is 0.

"scalar component" and "vector projection" Suppose $\theta \in [0, \pi)$ is the angle between two vectors $\mathbf{v}$ and $\mathbf{w} \neq \mathbf{0}$. The scalar component of $\mathbf{v}$ in the direction of $\mathbf{w}$ is

$$\text{comp}_{\mathbf{w}}\mathbf{v} = |\mathbf{v}|\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|}.$$

The vector projection of $\mathbf{v}$ onto $\mathbf{w}$ is

$$\text{proj}_{\mathbf{w}}\mathbf{v} = (\text{comp}_{\mathbf{w}}\mathbf{v})\frac{\mathbf{w}}{|\mathbf{w}|} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}.$$

proposition Suppose $\mathbf{v}, \mathbf{w} \neq \mathbf{0}$. Then

$$\mathbf{v} = \underbrace{\text{proj}_{\mathbf{w}}\mathbf{v}}_{\text{parallel to } \mathbf{w}} + \underbrace{(\mathbf{v} - \text{proj}_{\mathbf{w}}\mathbf{v})}_{\text{orthogonal to } \mathbf{w}}$$

"cross product" Suppose $\mathbf{v} = (v_1, v_2, v_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$, with an angle of θ between them. Then the cross product of $\mathbf{v}$ and $\mathbf{w}$ is defined to be the vector $\mathbf{x}$ such that

- $|\mathbf{x}| = |\mathbf{v}||\mathbf{w}|\sin\theta$.
- $\mathbf{x} \perp \mathbf{v}$ and $\mathbf{x} \perp \mathbf{w}$.
- $\mathbf{v}, \mathbf{w}, \mathbf{x}$ form a right-handed triple.

formula for the cross product

properties of the cross product

"triple scalar product"

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points, lines, planes, spheres

In the following, consider **p** to be a fixed point, **x** to be a variable point, and **a**, **b**, **v**, and **n** to be vectors.

line The line containing the point **p** in direction **v** has parametric equation

$$\mathbf{r}(t) = \mathbf{p} + t\mathbf{v},\ t \in \mathbb{R}$$

note

plane The plane containing the point **p** and parallel to **a** and **b** has parametric equation

$$\mathbf{R}(u, v) = \mathbf{p} + u\mathbf{a} + v\mathbf{b},\ (u, v) \in \mathbb{R}^2.$$

The plane through **p** with normal vector **n** has equation

$$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0.$$

sphere The sphere with center **c** and radius r has equation

$$|\mathbf{x} - \mathbf{c}| = r.$$

The **closed ball** with the same center and radius is defined by $|\mathbf{x} - \mathbf{c}| \leq r$. The **open ball** with the same center and radius is defined by $|\mathbf{x} - \mathbf{c}| < r$.

point/vector-valued functions

"vector-valued function" A "vector-valued function" is a function whose codomain is a vector space; that is, one whose values are vectors. Often is is more profitable to think of it as a "**point-valued**" function, though mathematically it is equivalent.

Suppose $f(t)$ and $g(t)$ are functions on an interval $I \subset \mathbb{R}$, Then $\mathbf{r}(t) = f[t)\mathbf{i} + g(t)\mathbf{j}$ is a function $I \to \mathbb{R}^2$.

Suppose $f(t)$, $g(t)$, $h(t)$ are functions on an interval $I \subset \mathbb{R}$, Then $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is a function $I \to \mathbb{R}^3$.

"limit" We say that $\mathbf{r}(t)$ has limit $\mathbf{L}$ at c, and write

$$\lim_{t \to c} \mathbf{r}(t) = \mathbf{L},$$

if

$$\forall \epsilon > 0 \; \exists \delta > 0 \text{ such that } |t - c| < \delta \Rightarrow |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

The definition of limit is equivalent to a description that is easier to check.

proposition Suppose $\mathbf{r}(t) = (r_1(t), \ldots, r_n(t))$ is a function $I \to \mathbb{R}^n$ and $\mathbf{L} = (L_1, \ldots, L_n)$. Then

$$\lim_{t \to c} \mathbf{r}(t) = \mathbf{L}$$

$$\Updownarrow$$

$$\lim_{t \to c} r_1(t) = L_1, \ldots, \lim_{t \to c} r_n(t) = L_n.$$

proof

"continuous" Suppose $\mathbf{r}(t)$ is a vector-valued function. We say that $\mathbf{r}(t)$ is continuous at c if

$$\lim_{t \to c} \mathbf{r}(t) = \mathbf{r}(c).$$

Concisely, (just as with scalar-valued functions) a functions is continuous at c if the <u>limit at c</u> equals the <u>value at c</u>.

"derivative" The derivative of the vector-valued function $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}.$$

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j}.$$

If this exists (that is, if f and g have derivatives at t), we say that $\mathbf{r}(t)$ is differentiable at t.

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

If this exists (that is, if f, g, and h have derivatives at t), we say that $\mathbf{r}(t)$ is differentiable at t.

properties

"indefinite integral" This is the set of all antiderivatives of $\mathbf{r}(t)$, denoted $\int \mathbf{r}(t)dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector.

"definite integral" Let the notation be as above. Then

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(b) - \mathbf{R}(a).$$

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motion

position, velocity, speed, acceleration If the position of an object at time t is given by $\mathbf{r}(t)$, then:

- the velocity of the object is $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$.
- the speed of the object is $|\mathbf{v}(t)| = |\frac{d\mathbf{r}}{dt}|$.
- the acceleration is given by $\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$.

arc length and distance The length of a path $\mathbf{r}(t), t \in [a, b]$ is

$$\int_a^b |\mathbf{r}'(t)|\, dt.$$

Equivalently, this is the distance traveled by an object whose position is given by $\mathbf{r}(t)$. We write

$$s(t) = \int_0^t |\mathbf{r}'(u)| du.$$

Note that $\frac{ds}{dt} = |\mathbf{r}'(t)|$, the speed.

integral of scalar function over a curve We also define similarly the integral of a scalar function f over a paramentric curve $\mathbf{r}$.

$$\boxed{\int_{\mathbf{r}} f\ ds}$$

notes

the Frenet Frame

unit tangent vector

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

So $\mathbf{T}$ is the direction of motion, while $|\mathbf{v}|$ is the speed.

curvature

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|$$

principal unit normal

$$\mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

So $\mathbf{N}$ is the direction of turning.

"osculating circle" The osculating circle is the circle tangent to a curve with the same curvature and same $\mathbf{N}$ at the point of tangency.

"radius of curvature" The radius of curvature is the radius of the osculating circle. It is

$$\rho = \frac{1}{\kappa}.$$

"unit binormal vector"

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}$$

"Frenet Frame" Since $|\mathbf{T}|$ is constant, then by construction $\mathbf{N} \perp \mathbf{T}$. Again by construction (and the properties of the cross product), $\mathbf{B} \perp \mathbf{T}$ and $\mathbf{B} \perp \mathbf{N}$. The triple of vectors $\mathbf{T}, \mathbf{N}, \mathbf{B}$ form an orthogonal right-handed system and are called the "**Frenet Frame**".

"torsion" We define the "torsion" to be

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.$$

proposition For any moving particle,

$$\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N},$$

where $a_{\mathbf{T}} = \frac{d}{dt}|\mathbf{v}|$ and $a_{\mathbf{N}} = \kappa|\mathbf{v}|^2$.
proof

proposition

$$a_{\mathbf{N}} = \sqrt{|\mathbf{a}|^2 - a_{\mathbf{T}}^2}$$

proof

proposition

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$$

proof

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functions of several variables

"open ball" The "open ball" of radius $r > 0$ and center $\mathbf{x}_0 \in \mathbb{R}^n$ is the set

$$B(\mathbf{x}_0, r) = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{x}_0| < r\}.$$

"interior point" A point $\mathbf{x}_0$ in a region $R \subset \mathbb{R}^n$ is an "interior" point of R if there is some $\epsilon > 0$ such that $B[\mathbf{x}_0, \epsilon) \subset R$.

"boundary point" A point $\mathbf{x}_0$ in a region $R \subset \mathbb{R}^n$ is a "boundary" point of R if for any $\epsilon > 0$, $B(\mathbf{x}_0, \epsilon)$ intersects both R and $\mathbb{R}^n \backslash R$.

"open", "closed" A region is "open" if it contains none of its boundary points. (Equivalently, all its points are interior points.) A region is "closed" if it contains all of its boundary points.

"function of n variables" This is a function

$$\begin{aligned} f : U &\to \mathbb{R} \\ \mathbf{x} &\mapsto f(\mathbf{x}), \end{aligned}$$

where $U \subseteq \mathbb{R}^n$, and $\mathbf{x} = (x_1, \ldots, x_n)$ being the "n variables". This function is often denoted just "$f(x_1, \ldots x_n)$".

"level surface" Suppose we are given $f(x_1 \ldots, x_n)$ as above, and $c \in \mathbb{R}$. Then the "level surface of level c" is the set $\{\mathbf{x} \in U \mid f(\mathbf{x}) = c\}$. This is called a "**level curve**" if $n = 2$.

"limit" Suppose $\mathbf{x}_0$ is a boundary or interior point of the function f. We say that f has "limit" L as $\mathbf{x}$ approaches $\mathbf{x}_0$ if $\forall \epsilon > 0$ $\exists \delta > 0$ such that

$$|\mathbf{x} - \mathbf{x}_0| < \delta \Rightarrow |f(\mathbf{x}) - L| < \epsilon.$$

We write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L.$$

"continuity" A function $f(\mathbf{x})$ is "continuous at the point $\mathbf{x}_0$ if

- f is defined at $\mathbf{x}_0$.
- $\lim_{\mathbf{x}\to x_0} f(\mathbf{x})$ exists.
- $\lim_{\mathbf{x}\to x_0} f(\mathbf{x}) = \mathbf{f}(x_0)$.

"partial derivative" Let $f(x_1, \ldots, x_n)$ be a funtion of several variables. Then the "partial derivative of f with respect to x_i at the point $(a_1, \ldots, a_n)$" is

$$\frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) = \lim_{h\to 0} \frac{f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{h},$$

provided the limit exists.

Clairaut's Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, f_{yx} are defined on an open region containing a point (a, b), and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

proof

"gradient" Suppose $f(x_1, \ldots, x_n)$ is a funtion of several variables. Then the "gradient" of f is

$$\nabla f(x_1, \ldots, x_n) = \left(\frac{df}{dx_1}, \ldots, \frac{df}{dx_n}\right).$$

Chain Rule Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a function of several variables, and $\mathbf{u} : \mathbb{R} \to \mathbb{R}^n$ is a point-valued function, then

$$\frac{d}{dt} f(\mathbf{u}(t)) = \nabla f(\mathbf{u}(t)) \cdot \mathbf{u}'(t).$$

formula for implicit differentiation Suppose that $F(x, y)$ is differentiable, and that $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$, we have

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

proof

“directional derivative Let $f : \mathbb{R}^n \to \mathbb{R}$ be function of several variables, and let $\mathbf{u} \in \mathbb{R}^n$ be a unit vector. Then define

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h}.$$

“differentiable” A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a}$ if $D_{\mathbf{u}} f(\mathbf{a})$ exists for all unit vectors $\mathbf{u} \in \mathbb{R}^n$.

theorem Let θ be the angle between ∇f and unit vector $\mathbf{u}$. Then

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos \theta.$$

proof

corollary ∇f is normal to the level surface of f.

linear approximation Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}$. The linear approximation to f at $\mathbf{x}_0$ is

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0).$$

proof

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extrema

proposition Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}$. If f has a local extremum at the point $\mathbf{a}$ in the interior of U, and the gradient of $\mathbf{a}$ exists in neighborhood of $\mathbf{a}$, then $\nabla f(\mathbf{a}) = \mathbf{0}$.

question What is the shortest distance from the point $(2, 0, -1)$ to the plane $3x + 2y + z = 4$?

"local maximum" A function $f(x, y)$ is said to have a local maximum at (a, b) if $f(a, b) \geq f(x, y)$ for all (x, y) near (a, b). (Precisely, this means that there is a disc centered at (a, b) such that $f(a, b) \geq f(x, y)$ for all (x, y) in the disc.)

"local minimum" A function $f(x, y)$ is said to have a local minimum at (a, b) if $f(a, b) \leq f(x, y)$ for all (x, y) near (a, b). (Precisely, this means that there is a disc centered at (a, b) such that $f(a, b) \leq f(x, y)$ for all (x, y) in the disc.)

discussion

"critical point" Suppose that the function $f(x, y)$ is such that $\nabla f(a, b)$ is $= \mathbf{0}$ or does not exist. Then we say that (a, b) is a critical point for f.

We state our results so far in the following theorem.

theorem Suppose that $f(x, y)$ has either a local maximum or local minimum at (a, b). Then (a, b) is a critical point of $f(x, y)$.

caution

contrast with case of functions of one variable

"Hessian" Suppose the second partial derivatives of $f(x, y)$ exist and are continuous near (a, b) (on a disc centered at (a, b)). The Hessian at (a, b) is defined to be

$$\begin{aligned} H(a, b) =& f_{xx}(a, b) f_{yy}(a, b) - (f_{xy}(a, b))^2 \\ =& f_{xx}(a, b) f_{yy}(a, b) - (f_{yx}(a, b))^2, \end{aligned}$$

the two expressions being equal by Clairaut's Theorem.

With the Hessian $H(x, y)$ in hand, we can use the following test for $f(x, y)$.

Second Derivative Test for 2-variable functions Suppose the second partial derivatives of $f(x, y)$ exist and are continuous near the critical point (a, b). Then if

- $H(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).
- $H(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).
- $H(a, b) < 0$, then f has a saddle point at (a, b).

note

note

example

example

answer to question

problem

problem

What if we want to find the extrema of a function subject to one or more constraints?

theorem: the method of "Lagrange multipliers" If the function $f(\mathbf{x})$ of n variables, subject to the constraints $g_1(\mathbf{x}) = 0, \ldots, g_k(\mathbf{x}) = 0$, has a local extremum at $\mathbf{a}$, then at $\mathbf{a}$,

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k,$$

where $\lambda_1, \ldots, \lambda_k$ are real numbers. (The method of Lagrange Multipliers gives us $n + k$ equations in $n + k$ unknowns. To solve these equations, which are linear in $\lambda_1, \ldots, \lambda_k$, it is usually easiest to eliminate $\lambda_1, \ldots, \lambda_k$ first.)
proof

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double integrals

"double Riemann Sum" Suppose the function $f(x, y)$ is defined on the rectangle $D = [a, b] \times [c, d]$. Let

$$\Delta x = \frac{b-a}{m}, \Delta y = \frac{d-c}{n},$$

$$x_i = a + i\Delta x, y_i = c + j\Delta y,$$

for

$$i = 0, \ldots, m, j = 0, \ldots n,$$

so that $(a = x_0, x_1, \ldots, x_m = b)$ and $(c = y_0, y_1, \ldots, y_n = d)$ are partitions of $[a, b]$ and $[c, d]$, respectively. This way, D is subdivided into rectangles

$$[x_{i-1}, x_i] \times [y_{j-1}, y_j],$$

each of area $\Delta A = \Delta x \Delta y$.

We choose any points

$$(x, y)^*_{ij} \in [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad i = 1, \ldots m, \quad j = 1, \ldots n.$$

Then the double Riemann Sum for f corresponding to this data is

$$R_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} f((x, y)^*_{ij})\Delta A$$

"double integral" Taking the limit as $m, n \to \infty$, we obtain the integral of $f(x, y)$ over D:

$$\begin{aligned}\iint_D f(x, y)dA &= \lim_{m,n\to\infty} R_{m,n} \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f((x, y)^*_{ij})\Delta A\end{aligned}$$

average value Suppose the function f is as before, and we want to find its average value on the region $D = [a, b] \times [c, d]$. Then, to estimate this, we subdivide the rectangle as before, and again choose sample points

$$(x, y)^*_{ij} \in [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad i = 1, \ldots m, \quad j = 1, \ldots n,$$

one in each sub-rectangle. Then we simply take the usual average of the function's values at these mn sample points:

$$\begin{aligned}&\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} f((x,y)_{ij}^{*})}{mn}\\ =&\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} f((x,y)_{ij}^{*})\Delta A}{mn\Delta A}\\ =&\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} f((x,y)_{ij}^{*})\Delta A}{(b-a)(d-c)}\end{aligned}$$

Taking the limit as $m, n \to \infty$, we obtain the average value of f on $[a, b] \times [c, d]$:

$$f_{\text{av}} = \frac{\iint_D f(x,y)dA}{(b-a)(d-c)}$$

properties of double integrals The following properties are obtained directly from the definition of the double integral via double Riemann Sums:

- $\displaystyle\iint\limits_D cf(x,y)dA = c\iint\limits_D f(x,y)dA$
- $\displaystyle\iint\limits_D (f(x,y)+g(x,y))dA = \iint\limits_D f(x,y)dA + \iint\limits_D g(x,y)dA$
- $\displaystyle f(x,y) \le g(x,y) \Rightarrow \iint\limits_D f(x,y)dA \le \iint\limits_D g(x,y)dA$

Fubini's Theorem

$$\boxed{\int_c^d \int_a^b f(x,y)dxdy = \iint\limits_D f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx}$$

proof

discussion

double integral over more general region Suppose D is the region bounded by $x = a$, $x = b$, $y = g_1(x)$, and $y = g_2(x)$. Then

$$\boxed{\iint\limits_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx}$$

Suppose instead that D is the region bounded by $y = c$, $y = d$, $x = g_1(y)$, and $x = g_2(y)$. Then

$$\boxed{\iint_D f(x,y)dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy}$$

polar Fubini's Theorem Suppose

$$P = [a,b] \times [\alpha,\beta] \subseteq [0,\infty) \times \mathbb{R}$$

is a region so that the function

$$\begin{aligned} T :& P \to \mathbb{R} \\ &(r,\theta) \mapsto (r\cos\theta, r\sin\theta) \end{aligned}$$

is injective. Suppose $f(x,y)$ is a function on the region $T(P) \subseteq \mathbb{R}^2$, Then

$$\begin{aligned} \iint_{T(P)} f(x,y)dA &= \int_{[\alpha,\beta]} \int_{[a,b]} f(T(r,\theta))rdrd\theta \\ &= \int_{[a,b]} \int_{[\alpha,\beta]} f(T(r,\theta))rd\theta dr. \end{aligned}$$

proof

double integral in polar coordinates over more general region

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triple integrals

discussion Suppose the function $f(x, y, z)$ is defined on the block $D = [a, b] \times [c, d] \times [r, s]$. Let

$$\Delta x = \frac{b-a}{\ell}, \qquad \Delta y = \frac{d-c}{m}, \qquad \Delta z = \frac{s-r}{n},$$

$$x_i = a + i\Delta x, \qquad y_i = c + j\Delta y, \qquad z_k = r + k\Delta z$$

for

$$i = 0, \ldots, \ell, \qquad j = 0, \ldots, m, \qquad k = 0, \ldots n,$$

so that

$$(a = x_0, x_1, \ldots, x_\ell = b), \quad (c = y_0, y_1, \ldots, y_m = d), \quad \text{and } (r = z_0, z_1, \ldots, z_n = s)$$

are partitions of $[a, b]$, $[c, d]$, and $[r, s]$ respectively. This way, D is subdivided into blocks

$$[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k],$$

each of volume $\Delta V = \Delta x \Delta y \Delta z$.

We choose any points

$$(x, y, z)^*_{ijk} \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k],$$

$$i = 1, \ldots \ell, \quad j = 1, \ldots m, \quad k = 1, \ldots, n.$$

Then the triple Riemann Sum for f corresponding to this data is

$$R_{\ell,m,n} = \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f((x, y, z)^*_{ijk}) \Delta V$$

"triple integral" Taking the limit as $\ell, m, n \to \infty$, we obtain the integral of $f(x, y, z)$ over D:

$$\iiint_D f(x, y, z) dV = \lim_{\ell,m,n\to\infty} R_{\ell,m,n}$$

$$= \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f((x, y, z)^*_{ijk}) \Delta V$$

average value Suppose the function f is as before, and we want to find its average value on the region $D = [a,b] \times [c,d] \times [r,s]$. Then, to estimate this, we subdivide the block as before, and again choose sample points

$$((x,y,z)^*_{ijk}) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k],$$

$$i = 1, \dots \ell, \quad j = 1, \dots m, \quad k = 1, \dots n,$$

one in each sub-block. Then we simply take the usual average of the function's values at these ℓmn sample points:

$$\begin{aligned} &\frac{\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f((x,y,z)^*_{ijk})}{\ell, m, n} \\ =&\frac{\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f((x,y,z)^*_{ijk})\Delta V}{\ell mn \Delta V} \\ =&\frac{\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f((x,y,z)^*_{ijk})\Delta V}{(b-a)(d-c)(s-r)} \end{aligned}$$

Taking the limit as $m, n \to \infty$, we obtain the average value of f on $[a,b] \times [c,d] \times [r,s]$:

$$f_{\text{av}} = \frac{\iiint_D f(x,y,z)dV}{(b-a)(d-c)(s-r)}$$

properties of triple integrals The following properties are obtained directly from the definition of the triple integral via triple Riemann Sums:

- $\displaystyle\iiint\limits_D cf(x,y,z)dV = c\iiint\limits_D f(x,y,z)dV$
- $\displaystyle\iiint\limits_D (f(x,y,z) + g(x,y,z))dV = \iiint\limits_D f(x,y,z)dV + \iiint\limits_D g(x,y,z)dV$
- $\displaystyle f(x,y,z) \le g(x,y,z) \Rightarrow \iiint\limits_D f(x,y,z)dV \le \iiint\limits_D g(x,y,z)dV$

iterated integration

$$\boxed{\iiint\limits_D f(x,y,z)dV = \int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx}$$

proof

By symmetry (as with double integration) we could integrate with respect to the 3 variables x, y, z in any order. The equality of the integrals with the 6 possible orders of integration would be a version of **Fubini's Theorem for triple integrals**. For brevity, we will not state it here.

triple integral over more general region Suppose D is the region bounded by $x = a$, $x = b$, $y = g_1(x)$, $y = g_2(x)$, $z = h_1(x, y)$ and $z = h_2(x, y)$. Then

$$\boxed{\iiint\limits_D f(x, y, z)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)dzdydx}$$

---

polar coordinates

"polar plane" The polar plane represents points in the plane by pairs (r, θ). The coordinate r is called the "directed distance", and θ is called the "directed angle".

discussion

symmetry

"polar coordinates" Polar coordinates consists of variables $(r, \theta, z) \in [0, \infty] \times [0, 2\pi] \times \mathbb{R}$, where, in terms of these coordinates, rectangular coordinates are given by

$$\boxed{\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \end{cases}}$$

In the corresponding Reimann sums,

$$\Delta A \approx r\ \Delta r\ \Delta\theta,$$

so that

$$\boxed{\text{“}\ dA = r\ dr\ d\theta\ \text{”}}$$

in a triple integral.

relations between Cartesian and polar coordinates For Cartesian coordinates (x, y) and polar coordinates (r, θ), we have

$$x^2 + y^2 = r^2,$$

$$\frac{y}{x} = \tan\theta.$$

area

arc length

cylindrical and spherical coordinates

discussion Just as with rectangular and polar coordinates, there is more than one way of choosing 3-dimensional coordinates, and corresponding ways of expressing ΔV.

"cylindrical coordinates" Cylindrical coordinates expresses two of the three dimensions in polar coordinates. Precisely, polar coordinates consists of variables $(r, \theta, z) \in [0, \infty] \times [0, 2\pi] \times \mathbb{R}$, where, in terms of these coordinates, rectangular coordinates are given by

$$\boxed{\begin{cases} x = & r\cos\theta \\ y = & r\sin\theta \\ z = & z. \end{cases}}$$

In the corresponding Reimann sums,

$$\Delta V \approx r\ \Delta r\ \Delta\theta\ \Delta z,$$

so that

$$\boxed{\text{“ } dV = r\ dr\ d\theta\ dz \text{ ”}}$$

in a triple integral.

"spherical coordinates" Spherical coordinates uses a radius in 3 dimensions and two angles identify the location of a point. Precisely, polar coordinates consists of variables $(\rho, \varphi, \theta) \in [0, \infty] \times [0, \pi] \times [0, 2\pi]$, where, in terms of these coordinates, rectangular coordinates are given by

$$\boxed{\begin{cases} x = & \rho\sin\varphi\cos\theta \\ y = & \rho\sin\varphi\sin\theta \\ z = & \rho\cos\varphi. \end{cases}}$$

In the corresponding Reimann sums,

$$\Delta V \approx \rho^2 \sin\varphi\ \Delta\rho\ \Delta\varphi\ \Delta\theta,$$

so that

$$\boxed{\text{“ } dV = \rho^2 \sin\varphi\ d\rho\ d\varphi\ d\theta \text{ ”}}$$

in a triple integral.

note

substitution

Jacobian determinant Suppose that $D, E \subset \mathbb{R}^n$, and that $T : D \to E$ is an bijective and differentiable function. Write $\mathbf{y} = T(\mathbf{x})$. Then the "Jacobian determinant" (or just "Jacobian") is the determinant of the $n \times n$ matrix whose (i, j) entry is $\frac{\partial y_i}{\partial x_j}(\mathbf{x})$, written

$$J(\mathbf{x}) = \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots x_n)} = \left| \left(\frac{\partial y_i}{\partial x_j}(x_1, \ldots, x_n) \right)_{i,j} \right|$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} (x_1, \ldots, x_n).$$

substitution Suppose that $D, E \subset \mathbb{R}^n$, and that

$$\begin{aligned} T : D &\to E \\ \mathbf{x} &\mapsto \mathbf{y} \end{aligned}$$

is an bijective and differentiable function. If $f(\mathbf{y})$ is a function on the domain E, then, in the n-dimensional Riemann sums, we find that $\Delta y_1 \cdots \Delta y_n$ on the domain E is corresponds approximately, under T, to $|J(\mathbf{x})| \Delta x_1 \cdots \Delta x_n$ on the domain D. We arrive at the following equality:

$$\int \cdots \int_E f(y_1, \ldots, y_n) dy_1 \cdots dy_n$$

$$= \int \cdots \int_D f(T(x_1, \ldots, x_n)) |J(x_1, \ldots, x_n)| dx_1 \cdots dx_n.$$

A convenient way to remember this is

$$\boxed{\text{``}dy_1 \cdots dy_n = \left| \frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots x_n)} \right| dx_1 \cdots dx_n\text{''}}.$$

example

example

example

example

---

parametric curves and surfaces

"parametric curve" A "parametric curve" is a function

$$\begin{aligned} \mathbf{r} : I &\to \mathbb{R}^n \\ t &\mapsto \mathbf{r}(t) \end{aligned}$$

where $I \subseteq \mathbb{R}$ is an interval.

"scalar stretch" $|\mathbf{r}'|$

"vector stretch" $\mathbf{r}'$

"parametric surface" A parametric surface is a function

$$\begin{aligned} \mathbf{R} : K &\to \mathbb{R}^n \\ (u, v) &\mapsto \mathbf{R}(u, v) \end{aligned}$$

where $K \subseteq \mathbb{R}^2$ is a region.

"scalar stretch" $|\mathbf{R}_u \times \mathbf{R}_v|$

"vector stretch" $\mathbf{R}_u \times \mathbf{R}_v$

scalar functions and vector fields

“scalar function” Recall that a scalar multivariable function is a function

$$f : D \to \mathbb{R},$$

where $D \subseteq \mathbb{R}^n$.

“vector field” A vector field is a function

$$\mathbf{F} : D \to \mathbb{R}^n,$$

where $D \subseteq \mathbb{R}^n$.

example

new kinds of integrals

In order to formulate the Theorems of Green, Gauss, and Stokes, we need to define some new kinds of integrals.

setting All closed surfaces will be parametrized with an outward orientation unless stated otherwise. All simple closed curves will be parametrized counter-clockwise unless stated otherwise.

Let $W \subset \mathbb{R}^3$ be a region of space with boundary ∂W a closed surface. Let $K \subset \mathbb{R}^2$ be a region with boundary ∂K a simple closed curve. Let $I \subset \mathbb{R}$ be an interval.

Let

$$\mathbf{r} : I \to \mathbb{R}^n$$

be a parametric curve and let

$$\mathbf{R} : K \to \mathbb{R}^n$$

be a parametric surface with boundary $\partial \mathbf{R} = \mathbf{R}(\partial K)$ a simple closed curve.

Write $\mathbf{t}$ for the unit direction vector of a curve. Write $\mathbf{n}$ be the outward-pointing unit vector normal to a simple closed plane curve $\mathbf{r}$. Write $\mathbf{N}$ for the outward-pointing unit vector normal to a closed surface $\mathbf{R}$.

Let f be a scalar function; let $\mathbf{F}$ be a vector field.

integral of scalar function over a curve

$$\boxed{\int_{\mathbf{r}} f \, ds}$$

"flow" of vector field along a curve

$$\boxed{\int_{\mathbf{r}} \mathbf{F} \cdot \mathbf{t} \, ds}$$

"circulation"

work

“flux” of vector field across a curve on a surface

$$\boxed{\int_{\mathbf{r}} \mathbf{F} \cdot \mathbf{n}\, ds}$$

integral of scalar function over a surface

$$\boxed{\iint\limits_{\mathbf{R}} f\ dS}$$

“flux” of vector field through a surface

$$\boxed{\iint\limits_{\mathbf{R}} \mathbf{F} \cdot \mathbf{N}\, dS}$$

---

divergence and curl

"divergence" If $F(\mathbf{x}) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ is a vector field, the "divergence" of $\mathbf{F}$ is the scalar function

$$\boxed{\mathrm{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

note

"curl" If $F(\mathbf{x}) = F_1(x,y,z)\mathbf{i} + F_2(x,y,z)\mathbf{j} + F_3(x,y,z)\mathbf{k}$ is a vector field, the "curl" of $\mathbf{F}$ is the vector field

$$\boxed{\mathrm{curl}\mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}}.$$

note

Green's, Gauss', and Stokes' Theorems

Green's Theorem

$$\boxed{\oint_{\partial K} \mathbf{F} \cdot \mathbf{n}\, ds = \iint\limits_{K} \text{div}\mathbf{F}\, dA}$$

Green's Theorem says roughly that the divergence of a vector field on a planar region K is equal to the flux of the vector field across the boundary ∂K of the region K.

Gauss' Theorem

$$\boxed{\oiint\limits_{\partial W} \mathbf{F} \cdot \mathbf{N}\, dS = \iiint\limits_{W} \text{div}\mathbf{F}\, dV}$$

Gauss' Theorem says roughly that the divergence of a vector field on a region of space W is equal to the flux of the vector field across the boundary ∂W of the region W.

Stokes' Theorem

$$\boxed{\oint_{\partial \mathbf{R}} \mathbf{F} \cdot \mathbf{t}\, ds = \iint\limits_{\mathbf{R}} \text{curl}\mathbf{F} \cdot \mathbf{N}\, dS}$$

Stokes' Theorem says roughly that the curl of a vector field on a surface $\mathbf{R}$ is equal to the circulation of the vector field around the boundary $\partial\mathbf{R}$ of the surface $\mathbf{R}$.

note If $\mathbf{R}$ is a region in the xy-plane, then $\mathbf{N} = \mathbf{k}$. The scalar stretch is 1 and the vector stretch is $\mathbf{k}$.

conservative vector fields

"potential function" If $\nabla f(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, then f is said to be a potential function for $\mathbf{F}$.

theorem Let $D \subseteq \mathbb{R}^n$ and let $\mathbf{F} : D \to \mathbb{R}^n$ be a vector field on D. Then the following are equivalent.

- For every closed loop $\mathbf{r}$ in D,

$$\oint_{\mathbf{r}} \mathbf{F} \cdot \mathbf{t}\; ds = 0.$$

- For any two paths $\mathbf{r}_1 : [a_1, b_1] \to D$ and $\mathbf{r}_2 : [a_2, b_2] \to D$ with the same endpoints (i.e. $\mathbf{r}_1(a_1) = \mathbf{r}_2(a_2)$ and $\mathbf{r}_1(b_1) = \mathbf{r}_2(b_2)$), then

$$\int_{\mathbf{r}_1} \mathbf{F} \cdot \mathbf{t}\; ds = \int_{\mathbf{r}_2} \mathbf{F} \cdot \mathbf{t}\; ds$$

- The vector field $\mathbf{F}$ has a potential.
- The vector field $\mathbf{F}$ is "**irrotational**"; that is, $\text{curl}\mathbf{F} = \mathbf{0}$.

proof

"conservative" vector field A vector field $\mathbf{F}$ satisfying the above criteria is called a "conservative" vector field.

examples, proofs, and additional discussion

examples This allows us to write

$$(2, -3, 4) = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k},$$

and

$$(0, 1, -4) = \mathbf{j} - 4\mathbf{k},$$

for example

point vs. vector An element of $\mathbb{R}^2$ or $\mathbb{R}^3$ can be considered a point or a vector. Just as with cardinal numbers and ordinal numbers, this is a case where the same abstraction is used for two slightly different concepts.

Suppose $P = \mathbf{p} = (p_1, p_2)$ and $Q = \mathbf{q} = (q_1, q_2)$ are points. Then the position vector equivalent to $\overrightarrow{PQ}$ is $\mathbf{q} - \mathbf{p} = (q_1 - p_1, q_2 - p_2)$.

fact (Here a is a scalar and $\mathbf{v}$ is a vector.)

$$|a\mathbf{v}| = |a||\mathbf{v}|$$

proof Apply the Law of Cosines to the vectors **v** and **w**, and use the above fact. □

properties of the dot product

- $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$
- $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

formula for the cross product If $\mathbf{v} = (v_1, v_2, v_3), \mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$, then

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_2 w_3 - v_3 w_2).$$

This is sometimes written

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix},$$

but we should remember that this is only a mnemonic device.

properties of the cross product

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- $(k\mathbf{a}) \times \mathbf{b} = k(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (k\mathbf{b})$
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

note The line through $\mathbf{r}_0$ and $\mathbf{r}_1$ has parametric equation

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1,\ t \in \mathbb{R}$$

The segment with endpoints $\mathbf{r}_0$ and $\mathbf{r}_1$ has parametric equation

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1,\ t \in [0,1]$$

"triple scalar product" The "triple scalar product" of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (in that order) is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}).$$

It gives the oriented volume of the parallelopiped spanned by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. (By "oriented" we mean that if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right-handed system, the triple scalar product will be positive; if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a left-handed system, the triple scalar product will be negative.

note

$$\begin{aligned} & \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \\ = & \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) \\ = & \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \\ = & -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) \\ = & -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) \\ = & -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) \end{aligned}$$

proof Suppose that

$$\lim_{t \to c} \mathbf{r}(t) = \mathbf{L}.$$

Suppose we are given any $\epsilon > 0$. Choose δ so that

$$|t - c| < \delta \Rightarrow |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

Then for $i = 1, \ldots, n$,

$$|t - c| < \delta \Rightarrow |r_i(t) - L_i| \leq |\mathbf{r}(t) - \mathbf{L}| < \epsilon.$$

We conclude that

$$\lim_{t \to c} r_i(t) = L_i$$

for $i = 1, \ldots, n$.

Conversely, suppose that

$$\lim_{t \to c} r_i(t) = L_i$$

for $i = 1, \ldots, n$. Suppose we are given any $\epsilon > 0$. For each of $i = 1, \ldots, n$, choose δ_i such that

$$|t - c| < \delta_i \Rightarrow |r_i(t) - L_i| < \frac{\epsilon}{\sqrt{n}}.$$

Let $\delta = \min\{\delta_1, \ldots, \delta_n\}$. Then

$$\begin{aligned} |t - c| < \delta \Rightarrow |\mathbf{r}(t) - \mathbf{L}| &= \left| \sqrt{(r_1(t) - L_i)^2 + \cdots + (r_n(t) - L_n)^2} \right| \\ &< \left| \sqrt{(\frac{\epsilon}{\sqrt{n}})^2 + \cdots + (\frac{\epsilon}{\sqrt{n}})^2} \right| \\ &= \epsilon. \end{aligned}$$

We conclude that

$$\lim_{t \to c} \mathbf{r}(t) = \mathbf{L}.$$

□

properties Let $\mathbf{q}$ and $\mathbf{r}$ be vector valued functions of t, $\mathbf{c}$ a constant vector, f a differentiable scalar function of t, and c a constant scalar. Then

- $\mathbf{c}' = 0$.
- $(c\mathbf{q})' = c\mathbf{q}'$.
- $(f\mathbf{q})' = f'\mathbf{q} + f\mathbf{q}'$ (Scalar Product Rule).
- $(\mathbf{q} \cdot \mathbf{r})' = \mathbf{q}' \cdot \mathbf{r} + \mathbf{q} \cdot \mathbf{r}'$ (Dot Product Rule).
- $(\mathbf{q} \times \mathbf{r})' = \mathbf{q}' \times \mathbf{r} + \mathbf{q} \times \mathbf{r}'$ (Cross Product Rule).
- $\frac{d}{dt}\mathbf{q}(f(t)) = f'(t)\mathbf{q}'(f(t))$ (Chain Rule).

integral of scalar function over a curve

$$\int_{\mathbf{r}} f\ ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$$

note The length of the path $\mathbf{r}(t)$ is

$$\int_{\mathbf{r}} 1\ ds = \int_a^b |\mathbf{r}'(t)|\ ds.$$

notes

- If $|\mathbf{r}(t)|$ is constant, then $\mathbf{v}(t) \perp \mathbf{r}(t)$.
  (To see this, differentiate $|\mathbf{r}(t)|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t)$.)
- If $|\mathbf{v}(t)|$ is constant, then $\mathbf{a}(t) \perp \mathbf{v}(t)$.
  (To see this, differentiate $|\mathbf{v}(t)|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t)$.)

proof We have

$$\begin{aligned}\mathbf{a} =& \frac{d\mathbf{v}}{dt} \\ =& \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) \\ =& \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\ =& \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) \\ =& \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \\ =& \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.\end{aligned}$$

□

proof This follows immediately from the preceding result. □

proof Calculate

$$\begin{aligned}|\mathbf{v}\times\mathbf{a}| =&|\mathbf{v}\times(a_{\mathbf{T}}\mathbf{T}+a_{\mathbf{N}}\mathbf{N})|\\ =&|\mathbf{v}\times a_{\mathbf{T}}\mathbf{T}+\mathbf{v}\times a_{\mathbf{N}}\mathbf{N}|\\ =&|\mathbf{0}+\mathbf{v}\times a_{\mathbf{N}}\mathbf{N}|\\ =&|\mathbf{v}||a_{\mathbf{N}}\mathbf{N}|\\ =&|\mathbf{v}|\kappa|\mathbf{v}|^2\\ =&\kappa|\mathbf{v}|^3.\end{aligned}$$

The result follows. □

discussion Whereas the Cartesian plane consists of points uniquely represented by elements of $\mathbb{R}^2$, the "polar plane" consists of points non-uniquely represented by elements of $\mathbb{R}^2$, subject to the equalities

$$(r, \theta) = (r, \theta + 2\pi),$$

$$(r, \theta) = (-r, \theta + \pi),$$

and

$$(0, \theta) = (0, 0).$$

In technical language, the polar plane is the co-image of the function

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R}^2 \\ (r, \theta) &\mapsto (r\cos\theta, r\sin\theta). \end{aligned}$$

The set of points $\{(r, 0) \nmid r \geq 0\}$ coincides with the positive horizontal axis $\{(x, 0) \nmid x \geq 0\}$ in the Cartesian plane. The directed angle θ is measured counter-clockwise from this ray.

symmetry A figure S in the polar plane is symmetric

- about the horizontal axis if $(r, \theta) \in S \Rightarrow (r, -\theta) \in S$.
- about the vertical axis if $(r, \theta) \in S \Rightarrow (r, \pi - \theta) \in S$.
- about the origin with n-fold rotational symmetry if $(r, \theta) \in S \Rightarrow (r, \theta + \frac{2\pi}{n}) \in S$.

area The area between the graph $r = f(\theta)$ and the origin, between $\theta = \alpha$ and $\theta - \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta.$$

arc length The length of the graph $r = f(\theta)$, $\alpha \leq \theta \leq \beta$ is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

The length of the graph $\theta = f(r)$, $a \leq r \leq b$ is

$$L = \int_{a}^{b} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2} dr.$$

proof Define the function

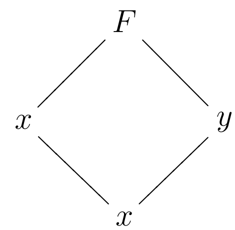
$$\Delta(h,j) = \frac{f(a+h,b+j) - f(a,b+j) - f(a+h,b) + f(a,b)}{hj}.$$

Then

$$\begin{aligned} f_{xy}(a,b) &= \lim_{j\to 0}\lim_{h\to 0}\Delta(h,j) \\ &= \lim_{(h,j)\to(0,0)}\Delta(h,j) \\ &= \lim_{h\to 0}\lim_{j\to 0}\Delta(h,j) = f_{yx}(a,b). \end{aligned}$$

□

proof Consider y as a function of x and differentiate $0 = F(x, y)$ with respect to x. Use the chain rule, according to the following diagram.



We have

$$0 = \frac{d}{dx}F = \frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = F_x + F_y \frac{dy}{dx},$$

$$\Rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}.$$

□

proof Define the parametric curve $\mathbf{r}(h) = \mathbf{x} + h\mathbf{u}$. Then

$$\begin{aligned}
D_{\mathbf{u}} f(\mathbf{x}) &= \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{u}) - f(\mathbf{x})}{h} \\
&= \lim_{h \to 0} \frac{f(\mathbf{r}(h)) - f(\mathbf{r}(0))}{h} \\
&= \frac{d}{dt} f(\mathbf{r}(t)) \bigg|_{t=0} \\
&= \nabla f(\mathbf{r}(0)) \cdot \mathbf{r}'(0) \text{ by the Chain Rule} \\
&= \nabla f(\mathbf{x}) \cdot \mathbf{u} \\
&= |\nabla f(\mathbf{x})||\mathbf{u}| \cos\theta \\
&= |\nabla f(\mathbf{x})| \cos\theta.
\end{aligned}$$

□

proof The vector giving the direction from $\mathbf{x}_0$ to $\mathbf{x}$ is $\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}$. Accordingly,

$$\begin{aligned} f(\mathbf{x}) \approx & f(\mathbf{x}_0) + \left(D_{\frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|}} f(\mathbf{x}_0) \right) |\mathbf{x}-\mathbf{x}_0| \\ = & f(\mathbf{x}_0) + \left(\nabla f(\mathbf{x}_0) \cdot \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|} \right) |\mathbf{x}-\mathbf{x}_0| \\ = & f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x}-\mathbf{x}_0). \end{aligned}$$

□

question What is the shortest distance from the point $(2, 0, -1)$ to the plane $3x + 2y + z = 4$?

first steps to a solution This shortest distance is going to be the distance from some point on the plane to $(2, 0, -1)$. Now the distance from any point (x, y, z) to $(2, 0, -1)$ is

$$d = \sqrt{(x-2)^2 + (y-0)^2 + (z-(-1))^2}.$$

We want to minimize d, and d will be a minimum exactly when its square,

$$d^2 = (x-2)^2 + (y-0)^2 + (z-(-1))^2,$$

is a minimum. The expression for d^2 is much simpler for taking derivatives. So we will try to find the point (x, y, z) on the plane such that d^2 is as small as possible. (This is a **common technique** in optimization problems.)

Now, if (x, y, z) lies on the plane $3x + 2y + z = 4$, then $z = 4 - 3x - 2y$. Substituting this into the expression for d^2 yields

$$(x-2)^2 + y^2 + (5 - 3x - 2y)^2,$$

a function of x and y that we will call $f(x, y)$. Now we need to find the minimum value of $f(x, y)$.

discussion Let us consider the situation, in general, of finding local maxima and minima of functions of two variables. Suppose that $f(x, y)$ has a local maximum at (a, b), and that both partial derivatives exist at (a, b). Let $g(x) = f(x, b)$ and $h(y) = f(a, y)$, functions of one variable. Then $g(x)$ has a local maximum at a, and $h(y)$ has a local maximum at b. From single-variable calculus, we know that $g'(a) = 0$ and $h'(b) = 0$. But $g'(a) = f_x(a, b)$ and $h'(b) = g_y(a, b)$. We conclude that if $f(x, y)$ has a local maximum at (a, b), then $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Similarly, we can show that if $f(x, y)$ has a local minimum at (a, b), then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

caution This does not mean that every critical point is a local maximum or minimum, but it does mean that such points are the only candidates for a local maximum or minimum.

contrast with case of functions of one variable At this point, after identifying critical points, we would, if f were a function of one variable, use the second derivative to try to discover whether the graph of f is concave up or concave down at each critical point. But for a function $f(x, y)$ of two variables, the graph may be concave up as (x, y) varies along one line in $\mathbb{R}^2$ through the point (a, b), and concave down as (x, y) varies along another such line. In this case (a, b) is called a **"saddle point"** of f, which is neither a local maximum nor a local minimum. What we need first, therefore, is something to tell us whether the graph curves the "same way" (concave up or down) as (x, y) varies along every line in $\mathbb{R}^2$ through (a, b).

note If $H(a, b) = 0$, the above test is inconclusive.

note We could just as well use $f_{yy}(a, b)$ in the above test in place of $f_{xx}(a, b)$.

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example The function $f(x, y) = x^2 + y^2 - 2x + 2y + 2$ has a local minimum at $(1, -1)$.

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example The function

$$f(x, y) = x^2 - y^2$$

has a saddle point at $(0, 0)$.

answer to question Now let us return to the question with which we started the discussion. We wish to find the miminum value of

$$f(x,y) = (x-2)^2 + y^2 + (5-3x-2y)^2.$$

We know that there is a closest point on the plane $3x+2y+z=4$ to the point $(2,0,-1)$, so we are assured of $f(x,y)$ having a minimum value at some point (a,b). Also, this value will be smaller than the value of $f(x,y)$ at any other point near (a,b), so that $f(x,y)$ will have a local minimum at (a,b). So we can use our Second Derivative Test to look for this point.

We have:

$$\begin{aligned} f_x(x,y) =& 2(x-2) - 6(5-3x-2y) = 20x + 12y - 34, \\ f_y(x,y) =& 2y - 4(5-3x-2y) = 12x + 10y - 20, \\ f_{xx}(x,y) =& 20, \\ H(x,y) =& f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 = 20 \cdot 10 - (12)^2 = 56. \end{aligned}$$

We find one critical point $(x,y) = (\frac{25}{14}, -\frac{1}{7})$, at which $H(x,y) > 0$ and $f_{xx}(x,y) > 0$. So $f(x,y)$ has a local minimum at $(\frac{25}{14}, -\frac{1}{7})$.

Using the equation of the plane, $3x+2y+z=4$, we find that the point on the plane having these values of x and y is $(\frac{25}{14}, -\frac{1}{7}, -\frac{15}{14})$. This is the point on the plane closest to the point $(2,0,-1)$. The distance between the two points is $\frac{\sqrt{14}}{14} = \frac{1}{\sqrt{14}}$. In conclusion, the desired distance from the point $(2,0,-1)$ to the plane $3x+2y+z=4$ is $\frac{1}{\sqrt{14}}$. $\square$

problem Find all local maxima, local minima, and saddle points of the function

$$f(x, y) = 4 + 4xy - x^4 - y^4.$$

solution First of all, we have:

$$\begin{aligned}
f_x(x, y) =& 4y - 4x^3, \\
f_y(x, y) =& 4x - 4y^3, \\
f_{xx}(x, y) =& -12x^2, \\
H(x, y) =& f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2 \\
=& (-12x^2)(-12y^2) - (4)^2 \\
=& 144x^2y^2 - 16.
\end{aligned}$$

We find exactly three critical points, at $(0, 0)$, $(1, 1)$, and $(-1, -1)$. Since $H(0, 0) = -16$, f has a saddle point at $(0, 0)$. At the other two points, $H(x, y) = 128$, and since $f_{xx}(x, y) = -12$ at both points, we find that $f(x, y)$ has a local maximum at both points $(1, 1)$ and $(-1, -1)$. □

problem We wish to make a rectangular container, with open top, from 12m^2 of steel. How large can we make the volume of the container, and what dimensions will give us that volume?

solution Let the length, width, and height of the box be x, y, z, respectively, measured in meters. The volume of the box is then $V = xyz$. Due to the amount of steel available, we know that

$$2xz + 2yz + xy = 24.$$

We can express z in terms of x and y:

$$z = \frac{(24 - xy)}{2(x + y)}.$$

Consequently,

$$V = xy\frac{(24 - xy)}{2(x + y)} = \frac{(24xy - x^2y^2)}{2(x + y)}.$$

Now $V(x, y)$ will be maximum at some point (x, y), and will clearly be defined near this point. Therefore, $V(x, y)$ will have a local maximum at this point, so we can look for critical points and apply the Second Derivative Test. The partial derivatives are

$$\frac{\partial V}{\partial x} = \frac{y^2(12 - 2xy - x^2)}{2(x + y)^2}$$

and

$$\frac{\partial V}{\partial y} = \frac{x^2(12 - 2xy - y^2)}{2(x + y)^2}$$

The only critical point turns out to be $(x, y) = (2, 2)$. The Second Derivative Test would show it to be a local maximum, but this would involve the quite cumbersome calculations of the second partial derivatives. In this case the easiest thing would be to note that there is a maximum, it is a local maximum, and the only possible local maximum is when $(x, y, z) = (2, 2, 1)$. These, therefore, must be the optimal dimensions. So the largest we can make the container is 4m^3. □

proof If $S \subseteq \mathbb{R}^n$ is the surface defined by $g_1(\mathbf{x}) = 0, \ldots, g_k(\mathbf{x}) = 0$, and $\mathbf{a} \in S$ is the point where the extremum occurs, then the directional derivative of $f(\mathbf{x})$ at $\mathbf{a}$ in all directions tangent to S are 0. It follows that $\nabla f(\mathbf{a})$ is orthogonal to S at $\mathbf{a}$. Now the tangent space of S at $\mathbf{a}$ is the intersection of the tangent spaces to the surfaces defined by each of $g_1(\mathbf{x}) = 0, \ldots, g_k(\mathbf{x}) = 0$, which are the spaces $\nabla g_1(\mathbf{a})^\perp, \ldots, \nabla g_k(\mathbf{x})^\perp$. This intersection is the space spanned by $\nabla g_1(\mathbf{a}), \ldots, \nabla g_k(\mathbf{a})$. Consequently, $\nabla f(\mathbf{a})$ lies in the span of $\nabla g_1(\mathbf{a}), \ldots, \nabla g_k(\mathbf{x})$. $\square$

proof Suppose we choose

$$x_i^* \in [x_{i-1}, x_i], \quad i = 1, \dots m,$$

$$y_j^* \in [y_{j-1}, y_j], \quad j = 1, \dots n$$

and set

$$x_{ij}^* = x_i^* \quad j = 1, \dots n,$$
$$y_{ij}^* = y_j^* \quad i = 1, \dots m.$$

(This way, all the points (x_{ij}^*, y_{ij}^*) are "aligned" vertically and horizontally.)

The double Riemann Sum for f corresponding to this data is

$$\begin{aligned} R_{m,n} &= \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A \\ &= \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta y \Delta x \end{aligned}$$

Taking the limit of $R_{m,n}$ as $m, n \to \infty$, we obtain

$$\begin{aligned} \iint_D f(x,y) dA &= \lim_{m,n\to\infty} R_{m,n} \\ &= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta y \Delta x \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta y \Delta x \\ &= \lim_{m\to\infty} \sum_{i=1}^{m} \left(\lim_{n\to\infty} \sum_{j=1}^{n} f(x_i^*, y_j^*) \Delta y \right) \Delta x \\ &= \lim_{m\to\infty} \sum_{i=1}^{m} \left(\int_{[c,d]} f(x_i^*, y) dy \right) \Delta x \\ &= \int_{[a,b]} \left(\int_{[c,d]} f(x,y) dy \right) dx \\ &= \int_a^b \int_c^d f(x,y) dy dx. \end{aligned}$$

This proves one of the equalities. The other is shown in exactly the same way. □

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discussion Suppose the function $f(x, y)$ is defined on the region $D \subseteq [a, b] \times [c, d]$. Then we define a function $\tilde{f}(x, y)$ on the rectangle $[a, b] \times [c, d]$ by

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in R \\ 0 & \text{if } (x, y) \notin R \end{cases}$$

double integral over general region We define

$$\iint\limits_{R} f(x, y)dA = \iint\limits_{[a,b]\times[c,d]} \tilde{f}(x, y)dA.$$

special cases Suppose R is the region bounded by $x = a$, $x = b$, $y = g_1(x)$, and $y = g_2(x)$, with $c \leq g_1(x) \leq g_2(x) \leq d$. Then,

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } g_1(x) \leq y \leq g_2(x) \\ 0 & \text{otherwise} \end{cases}$$

In this case we have

$$\begin{aligned} \iint\limits_{R} f(x, y)dA &= \iint\limits_{[a,b]\times[c,d]} \tilde{f}(x, y)dA \\ &= \int_a^b \int_c^d \tilde{f}(x, y)dydx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx. \end{aligned}$$

The same argument with x and y switched shows that if R is the region bounded by $y = c$, $y = d$, $x = g_1(y)$, and $x = g_2(y)$, with $a \leq g_1(y) \leq g_2(y) \leq b$, then

$$\begin{aligned} \iint\limits_{R} f(x, y)dA &= \iint\limits_{[a,b]\times[c,d]} \tilde{f}(x, y)dA \\ &= \int_c^d \int_a^b \tilde{f}(x, y)dxdy \\ &= \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)dxdy. \end{aligned}$$

proof We aim to construct a Riemann sum in polar coordinates (r, θ). We subdivide $T(P)$ into sub-regions as follows. Let

$$\Delta r = \frac{r_2 - r_1}{m}, \quad \Delta\theta = \frac{\alpha_2 - \beta_1}{n},$$

and

$$r_i = a + i\Delta r, \quad \theta_j = \alpha + j\Delta\theta$$

Then P is sub-divided into regions

$$T([r_{i-1}, r_i] \times [\theta_{j-1} \times \theta_j]), \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$

Choose points

$$(r^*_{ij}, \theta^*_{ij}) \in [r_{i-1}, r_i] \times [\theta_{j-1} \times \theta_j], \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$

Then, writing $(x, y) = T(r, \theta)$, we have

$$\Delta A \approx (\Delta r)\left(\frac{\Delta\theta}{2\pi} 2\pi r\right) = r(\Delta r)(\Delta\theta).$$

The Riemann Sum is

$$R_{m,n} = \sum_{j=1}^{n} \sum_{i=1}^{m} f(T(r_{ij}, \theta_{ij})) r_{ij} \Delta r \Delta\theta,$$

which has the limit, as $m, n \to \infty$, of

$$\begin{aligned} \iint\limits_{T(P)} f(x, y) dA &= \int_{[\alpha,\beta]} \int_{[a,b]} f(T(r, \theta)) r dr d\theta \\ &= \int_{[a,b]} \int_{[\alpha,\beta]} f(T(r, \theta)) r d\theta dr \end{aligned}$$

(the second equality coming from similar considerations). □

double integral in polar coordinates over more general region As with rectangular coordinates, suppose now that P is bounded by $r = a$, $r = b$, $\theta = g_1(r)$, and $\theta = g_2(r)$, where $g_1(r) \leq g_2(r)$. Then

$$\iint_{T(P)} f(x,y)dA = \int_{[a,b]} \int_{[g_1(r),g_2(r)]} f(T(r,\theta))rd\theta dr.$$

On the other hand, if P is bounded by $\theta = \alpha$, $\theta = \beta$, $r = g_1(\theta)$, and $r = g_2(\theta)$, where $g_1(\theta) \leq g_2(\theta)$, then

$$\iint_{T(P)} f(x,y)dA = \int_{[\alpha,\beta]} \int_{[g_1(\theta),g_2(\theta)]} f(T(r,\theta))rdrd\theta.$$

□

proof Suppose we choose

$$x_i^* \in [x_{i-1}, x_i], \quad i = 1, \dots \ell,$$

$$y_j^* \in [y_{j-1}, y_j], \quad j = 1, \dots m$$

$$z_k^* \in [z_{k-1}, z_k], \quad k = 1, \dots n$$

and set

$$x_{ijk}^* = x_i^* \quad i = 1, \dots \ell,$$

$$y_{ijk}^* = y_j^* \quad j = 1, \dots m.$$

$$z_{ijk}^* = z_k^* \quad k = 1, \dots n.$$

(This way, all the points $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ are "aligned" in all three dimensions.) 3 Then the triple Riemann Sum for f corresponding to this data is

$$R_{\ell,m,n} = \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*) \Delta z \Delta y \Delta x.$$

Taking the limit as $\ell, m, n \to \infty$, we obtain

$$\begin{aligned}
\iiint_D f(x,y,z)dV &= \lim_{\ell,m,n\to\infty} R_{\ell,m,n} \\
&= \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\Delta z \Delta y \Delta x \\
&= \lim_{\ell\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\Delta z \Delta y \Delta x \\
&= \lim_{\ell\to\infty} \sum_{i=1}^{\ell} \left(\lim_{m\to\infty} \sum_{j=1}^{m} \left(\lim_{n\to\infty} \sum_{k=1}^{n} f(x_i^*, y_j^*, z_k^*)\Delta z \right) \Delta y \right) \Delta x \\
&= \lim_{\ell\to\infty} \sum_{i=1}^{\ell} \left(\lim_{m\to\infty} \sum_{j=1}^{m} \left(\int_{[r,s]} f(x_i^*, y_j^*, z)dz \right) \Delta y \right) \Delta x \\
&= \lim_{\ell\to\infty} \sum_{i=1}^{\ell} \left(\int_{[c,d]} \left(\int_{[r,s]} f(x_i^*, y, z)dz \right) dy \right) \Delta x \\
&= \int_{[a,b]} \left(\int_{[c,d]} \left(\int_{[r,s]} f(x,y,z)dz \right) dy \right) dx \\
&= \int_a^b \int_c^d \int_r^s f(x,y,z)dzdydx
\end{aligned}$$

By symmetry (as with double integration) we could integrate with respect to the 3 variables x, y, z in any order. The equality of the integrals with the 6 possible orders of integration would be a version of **Fubini's Theorem for triple integrals**. For brevity, we will not state it here.

---

note The conversion from cylindrical to spherical coordinates is given by

$$\boxed{\begin{cases} r = & \rho\sin\varphi \\ \theta = & \theta \\ z = & \rho\cos\varphi. \end{cases}}$$

In the corresponding Reimann sums,

$$r\Delta r\Delta\theta\Delta z \approx \Delta V \approx \rho^2\sin\varphi\Delta\rho\Delta\varphi\Delta\theta,$$

so that

$$\boxed{\text{“ } drd\theta dz = \rho d\rho d\varphi d\theta \text{ ”}}$$

in a triple integral.

example In changing from rectangular to polar coordinates,

$$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = r.$$

example In changing from rectangular to cylindrical coordinates,

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r.$$

example In changing from rectangular to spherical coordinates,

$$\left|\frac{\partial(x, y, z)}{\partial(\rho, \varphi, \theta)}\right| = \rho^2 \sin\varphi.$$

example In changing from cylindrical to spherical coordinates,

$$\left|\frac{\partial(r,\theta,z)}{\partial(\rho,\varphi,\theta)}\right| = \rho.$$

example The gradient ∇f of a multivariable function f is a vector field.

integral of scalar function over a curve

$$\int_{\mathbf{r}} f \, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$$

“flow” of vector field along a curve

$$\int_{\mathbf{r}} \mathbf{F} \cdot \mathbf{t} \, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

“circulation” Suppose that $\mathbf{r} : [a, b] \to D \subseteq \mathbb{R}^2$ is a closed loop – that is, such that $\mathbf{r}(a) = \mathbf{r}(b)$ – parametrized counter-clockwise. Suppose $\mathbf{F} : D \to \mathbb{R}^2$ is a vector field. Then

$$\oint_{\mathbf{r}} \mathbf{F} \cdot \mathbf{t} \, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt$$

is called the “circulation” of $\mathbf{F}$ around $\mathbf{r}$.

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work The work done by a force field $\mathbf{F}$ on a particle whose position is given by $\mathbf{r}$ is

$$W = \int_{\mathbf{r}} \mathbf{F} \cdot \mathbf{t} \, ds.$$

“flux” of vector field across a curve

$$\int_{\mathbf{r}} \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{n} |\mathbf{r}'(t)|\, dt$$

formula for flux across a closed curve Suppose that $\mathbf{r} = r_1\mathbf{i} + r_2\mathbf{j} : [a, b] \to D \subseteq \mathbb{R}^2$ is a closed loop – that is, such that $\mathbf{r}(a) = \mathbf{r}(b)$ – parametrized counter-clockwise. Then

$$\mathbf{n}(t) = \mathbf{t}(t) \times \mathbf{k} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \times \mathbf{k} = \frac{r_1'(t)\mathbf{i} + r_2'(t)\mathbf{j}}{|\mathbf{r}'(t)|} \times \mathbf{k} = \frac{r_2'(t)\mathbf{i} - r_1'(t)\mathbf{j}}{|\mathbf{r}'(t)|}$$

is the outward-pointing normal, so that the flux is

$$\oint_{\mathbf{r}} \mathbf{F} \cdot \mathbf{n}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t) |\mathbf{r}'(t)|\, dt = \int_a^b \mathbf{F} \cdot (r_2', -r_1')\, dt.$$

integral of scalar function over a surface

$$\iint_{\mathbf{R}} f\ dS = \iint_{D} f(\mathbf{R}(u,v))|\mathbf{R}_u \times \mathbf{R}_v| du dv$$

“flux” of vector field through a surface

$$\iint_{\mathbf{R}} \mathbf{F} \cdot \mathbf{N} \, dS = \iint_{D} \mathbf{F}(\mathbf{R}(u,v)) \cdot (\mathbf{R}_u \times \mathbf{R}_v) du dv$$

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note This is sometimes remembered by the convenient (though imprecise) notation "$\nabla \cdot \mathbf{F}$". If $\mathbf{F}$ is a 2-dimensional vector field in the plane, we simply construct a 3rd dimension and let $F_3 = 0$.

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note This is sometimes remembered by the convenient (though imprecise) notation "$\nabla \times \mathbf{F}$". If $\mathbf{F}$ is a 2-dimensional vector field in the plane, we simply construct a 3rd dimension and let $F_3 = 0$.

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proof We prove that the four statements are equivalent in four steps: we prove that the first implies the second, that the second implies the third, that the third implies the fourth, and finally, that the fourth implies the first.

Assume that the first statement is true. Suppose

$$\mathbf{r}_1 : [a_1, b_1] \to D$$

and

$$\mathbf{r}_2 : [a_2, b_2] \to D$$

are two paths with the same endpoints (i.e. $\mathbf{r}_1(a_1) = \mathbf{r}_2(a_2)$ and $\mathbf{r}_1(a_2) = \mathbf{r}_2(b_2)$). Construct

$$\begin{aligned} \mathbf{c} &: [-1, 1] \to D \\ t &\to \begin{cases} \mathbf{r}_1\left(b_1 + t(b_1 - a_1)\right) & \text{if } t \in [-1, 0] \\ \mathbf{r}_2\left(b_2 + t(a_2 - b_2)\right) & \text{if } t \in [0, 1], \end{cases} \end{aligned}$$

the path $\mathbf{r}_1$ followed by the reverse of $\mathbf{r}_2$. Then

$$\int_{\mathbf{r}_1} \mathbf{F} \cdot d\mathbf{s} - \int_{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = 0$$

since $\mathbf{c}$ is a closed path. Thus the second statement is true.

Now assume that the second statement is true. Fix $\mathbf{x}_0 \in D$. For any $\mathbf{x} \in D$, define

$$f(\mathbf{x}) = \int_{\mathbf{r}} \mathbf{F} \cdot \mathbf{t}\, ds,$$

where $\mathbf{r} : [a, b] \to D$ is any curve with $\mathbf{r}(a) = \mathbf{x}_0$ and $\mathbf{r}(b) = \mathbf{x}$. (This is well-defined since the integral does not depend on our choice of $\mathbf{r}$, by assumption.) Let $\mathbf{x} \in D$. Let $\mathbf{u}$ be any unit vector. Choose $\mathbf{r} : [a, b + \epsilon] \to D$ such that

$\mathbf{r}(a) = \mathbf{x}_0$, $\mathbf{r}(b) = \mathbf{x}$, and $\mathbf{r}'(b) = \mathbf{u}$. Then

$$\begin{aligned}
\nabla f(\mathbf{x}) \cdot \mathbf{u} =& \nabla f(\mathbf{r}(b)) \cdot \mathbf{r}'(b) \\
=& \frac{d}{dh} f(\mathbf{r}(b+h)) \bigg|_{h=0} \\
& \text{(by the Chain Rule)} \\
=& \frac{d}{dh} \left(\int_a^{b+h} \mathbf{F} \cdot \mathbf{t} \, dt \right) \bigg|_{h=0} \\
& \text{(since } \mathbf{r}|_{[a,b+h]} \text{ is a path from } \mathbf{x}_0 \text{ to } \mathbf{r}(b+h)) \\
=& \mathbf{F}(\mathbf{r}(b+0)) \cdot \mathbf{t}(b+0) \\
& \text{(by the Fundamental Theorem Of Calculus)} \\
=& \mathbf{F}(\mathbf{r}(b)) \cdot \frac{\mathbf{r}'(b)}{|\mathbf{r}'(b)|} \\
=& \mathbf{F}(\mathbf{x}) \cdot \frac{\mathbf{u}}{1} \\
=& \mathbf{F}(\mathbf{x}) \cdot \mathbf{u}.
\end{aligned}$$

Since this is true for any unit vector $\mathbf{u}$, we conclude that $\nabla f(\mathbf{x}) = \mathbf{F}(\mathbf{x})$, so that f is a potential for the vector field $\mathbf{F}$. Thus the <u>third statement</u> is true.

Now assume that the <u>third statement</u> is true. Let f be a potential for the vector field $\mathbf{F}$. Direct application of the definitions of ∇ and curl show that

$$\text{curl}\mathbf{F} = \text{curl}(\nabla f) = \mathbf{0}.$$

Thus the <u>fourth statment</u> is true.

Now assume that the <u>fourth statement</u> is true. Let $\mathbf{r}$ be a simple closed loop in D. Let $\mathbf{R} \subseteq D$ be a surface such that $\partial\mathbf{R} = \mathbf{r}$. Then by Stokes' Theorem,

$$\oint_{\mathbf{r}} \mathbf{F} \cdot \mathbf{t} \, ds = \oint_{\partial\mathbf{R}} \mathbf{F} \cdot \mathbf{t} \, ds = \iint_{\mathbf{R}} \text{curl}\mathbf{F} \cdot \mathbf{N} \, dS = \iint_{\mathbf{R}} \mathbf{0} \cdot \mathbf{N} \, dS = 0.$$

Thus the <u>first statement</u> is true.

We have shown that the four statements are equivalent. □